Homework 1
Representation Theory

Joshua Ruiter

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Proposition 0.1 (Exercise 1)**.** *Let G be a group, and let V, W be finite-dimensional representations of G. Let $\mathrm{Hom}_G(V, W)$ be the set of linear maps $\phi : V \to W$ such that the following square commutes for every $g \in G$.*

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{\phi} & W \end{array}$$

The space $\mathrm{Hom}(V, W)$ is a representation of G via the isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$. Using this G-module structure, we can define

$$\mathrm{Hom}(V, W)^G = \{\phi \in \mathrm{Hom}(V, W) | g \cdot \phi = \phi\}$$

Then $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$.

Proof. We'll write $g\phi$ for composition and $g \cdot \phi$ for the action of g on $\phi \in \mathrm{Hom}(V, W)$ as a G-module in order to distinguish these notions. Note that

$$(g \cdot \phi)(v) = g\phi(g^{-1}v)$$

by the discussion on page 4 of Fulton & Harris. Let $\phi \in \mathrm{Hom}_G(V, W)$. Then since $g\phi = \phi g$, we get

$$(g \cdot \phi)(v) = g\phi(g^{-1}v) = \phi(gg^{-1}v) = \phi(v)$$

Thus $g \cdot \phi = \phi$, and $\phi \in \mathrm{Hom}(V, W)^G$. Thus $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)^G$. Now suppose $\phi \in \mathrm{Hom}(V, W)^G$. Then for all $v \in V$ and $g \in G$,

$$\phi(v) = (g^{-1} \cdot \phi)(v) = g^{-1}\phi(gv) \implies g\phi(v) = \phi(gv) \implies g\phi = \phi g$$

Thus $\phi \in \mathrm{Hom}_G(V, W)$, so we have the opposite containment $\mathrm{Hom}(V, W)^G \subset \mathrm{Hom}_G(V, W)$. Hence these are equal. $\square$

Proposition 0.2 (Exercise 2)**.** *Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G where $\dim V = n$, such that $\det \rho(g) = 1$ for all $g \in G$. Then $\wedge^k V$ and $\wedge^{n-k} V^*$ are isomorphic as representations of G.*

Proof. As is typical when dealing with representations, we'll be somewhat careless in our notation, and refer to the map $\rho(g)$ simply by g. It makes notation more compact, and hopefully does not cause too much confusion.

First, we claim that $\wedge^n V$ is the trivial representation, that is, $g \in G$ acts as the identity on $\wedge^n V$. Let $g \in G$, and choose a basis of V consisting of eigenvectors $w_1, \ldots, w_n$ for g. (We can do this because V is a $\mathbb{C}$-vector space.) Let λ_i be the corresponding eigenvector for w_i. Then $\wedge^n V$ is spanned by $w_1 \wedge \ldots \wedge w_n$, so it is sufficient to check that g acts as identity on this single spanning element. We compute how g acts on this element:

$$g(w_1 \wedge \ldots \wedge w_n) = gw_1 \wedge \ldots \wedge gw_n = \lambda_1 w_1 \wedge \ldots \wedge \lambda_n w_n = \left(\prod_{i=1}^{n} \lambda_i\right)(w_1 \wedge \ldots \wedge w_n)$$

By the hypothesis, the determinant of g is 1, so the product of eigenvectors is one. Hence $g(w_1 \wedge \ldots \wedge w_n) = w_1 \wedge \ldots \wedge w_n$, so g acts as the identity, as claimed. Thus $\wedge^n V$ is the trivial representation, that is, $\wedge^n V \cong \mathbb{C}$ as representations of G. Now consider the G-linear map

$$\langle , \rangle : \wedge^k V \times \wedge^{n-k} V \to \wedge^n V$$
$$\langle v_1 \wedge \ldots \wedge v_k, v_{k+1} \wedge \ldots \wedge v_n \rangle = v_1 \wedge \ldots \wedge v_k \wedge v_{k+1} \wedge \ldots \wedge v_n$$

This induces a linear map $\phi : \wedge^k V \to \operatorname{Hom}_G(\wedge^{n-k} V, \wedge^n V)$ defined by $\phi(x)(y) = \langle x, y \rangle$. Note that $\wedge^{n-k} V^* \cong (\wedge^{n-k} V)^* = \operatorname{Hom}_G(\wedge^{n-k} V, \mathbb{C})$ (see Appendix B.3, page 476 of Fulton & Harris). By our earlier remarks concerning $\wedge^n V$, we have $\wedge^{n-k} V^* \cong \operatorname{Hom}_G(\wedge^{n-k} V, \wedge^n V)$. Thus we can think of ϕ as a map $\wedge^k V \to \wedge^{n-k} V^*$, so if we can show that ϕ is an isomorphism we are done. Since the domain and range of ϕ are finite-dimensional vector spaces, by the Rank-Nullity Theorem injectivity of ϕ implies surjectivity, so it is sufficient to show that ϕ is injective.

Suppose $x \in \ker \phi$. That is, $\phi(x)(y) = \langle x, y \rangle = 0$ for all $y \in \wedge^{n-k} V$. Fix a basis $e_1, \ldots e_n$ of V. Then, using the notation of multi-indices, we can write x uniquely as $\sum_I a_I e_I$ where

$$I = (i_1, \ldots, i_k) \qquad 1 \le i_1 < i_2 < \ldots < i_k \le n \qquad e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$$

Suppose $x \neq 0$. Then choose I_0 so that $a_{I_0} \neq 0$, and let J_0 be the "complimentary" multi-index to I_0, that is, $J_0 = (j_{k+1}, \ldots, j_n)$ where $1 \le j_{k+1} < j_{k+2} < \ldots < j_n \le n$ and

$$\{i_1, \ldots, i_{k+1}\} \cap \{j_{k+1} \ldots, j_n\} = \emptyset \qquad \{i_1, \ldots, i_{k+1}\} \cup \{j_{k+1}, \ldots, j_n\} = \{1, \ldots, n\}$$

Define $y = e_{J_0}$. Then

$$\phi(x)(y) = \langle x, y \rangle = x \wedge y = \left(\sum_I a_I e_I\right) \wedge e_{J_0} = \sum_I a_I (e_I \wedge e_{J_0})$$

For $I \neq I_0$, the wedge product $e_I \wedge e_{J_0}$ will be zero, since there will be a repeated basis vector e_i. Thus

$$\phi(x)(y) = \sum_I a_I (e_I \wedge e_{J_0}) = a_{I_0}(e_{I_0} \wedge e_{J_0})$$

Since I_0 and J_0 were complimentary/disjoint, $e_{I_0} \wedge e_{J_0} = \pm(e_1 \wedge \ldots \wedge e_n)$, where the $\pm$ comes from the sign of the permutation required to get to the increasing order. Thus $\phi(x)(y) \neq 0$, which is a contradiction, so we conclude that $x = 0$ so $\ker \phi = 0$, and ϕ is injective, and hence it is an isomorphism. □

Lemma 0.3 (for Exercise 3a)**.** *Let G be a finite group, and let R_G be the regular representation (over $\mathbb{C}$), and let χ_{R_G} be the character. Then*

$$\chi_{R_G}(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

Proof. If $g = e$, then g acts as the identity on R_G, so its matrix representation is the identity matrix, of size $\dim R_G = |G|$. This has trace $|G|$. If $g \neq e$, we know that g acts to permute the basis of R_G, so g is represented by a permutation matrix (one 1 in each row and column, zeroes else). We know that g acting by left multiplication has no fixed points, because a fixed point h implies

$$gh = h \implies g = e$$

Thus g fixes none of the basis of R_G. Thus the diagonal entries of the matrix representation of g are all zero, so the $\operatorname{tr}(g) = \chi_{R_G}(g) = 0$. □

Proposition 0.4 (Exercise 3a)**.** *The regular representation R of S_3 decomposes as*

$$R \cong U \oplus U' \oplus V \oplus V$$

Proof. By the previous lemma, we know that the character of the regular represenation of S_3 is

$$\chi_R(g) = \begin{cases} 0 & g \neq e \\ 6 & g = e \end{cases}$$

A character table for S_3 can be found on page 14 of Fulton & Harris. We can write χ_R as the sum $\chi_R = \chi_U + \chi_{U'} + 2\chi_V$. Since a representation is determined by its character and this is also the character of

$$U \oplus U' \oplus V \oplus V$$

we get the isomorphism (of S_3-representations)

$$R \cong U \oplus U' \oplus V \oplus V$$

□

Lemma 0.5 (for Exercise 3b)**.** *Let V be the standard, irreducible, 2-dimensional representation of S_3. Then the character of $\operatorname{Sym}^k V$ is*

$$\chi_{\operatorname{Sym}^k V}(1) = k + 1$$

$$\chi_{\operatorname{Sym}^k V}(\sigma) = \begin{cases} 1 & k \equiv 0 \bmod 2 \\ 0 & k \equiv 1 \bmod 2 \end{cases}$$

$$\chi_{\operatorname{Sym}^k V}(\tau) = \begin{cases} 1 & k \equiv 0 \bmod 3 \\ -1 & k \equiv 1 \bmod 3 \\ 0 & k \equiv 2 \bmod 3 \end{cases}$$

where $\sigma = (1\ 2)$ and $\tau = (1\ 2\ 3)$ and $\omega = e^{2\pi i/3}$.

Proof. First, recall the usual basis e_1, e_2 of V with

$$\tau e_1 = \omega e_1 \qquad \tau e_2 = \omega^2 e_2 \qquad \sigma e_1 = e_2 \qquad \sigma e_2 = e_1$$

where $\tau = (1\ 2\ 3)$ and $\sigma = (1\ 2)$ and $\omega = e^{2\pi i/3}$. The usual basis for $\operatorname{Sym}^k V$ is given by

$$\{e_1 \cdot \ldots \cdot e_1, e_1 \cdot \ldots \cdot e_1 \cdot e_2, \ldots e_2 \cdot \ldots e_2\}$$

Note that $\dim \operatorname{Sym}^k V = k+1$, which gives us $\chi_{\operatorname{Sym}^k V}(1) = k+1$. We introduce the notation $v_{(i,j)} = e_1 \cdot \ldots \cdot e_1 \cdot e_2 \cdot \ldots \cdot e_2$ where e_1 appears i times and e_2 appears j times. Then we can rewrite the basis of $\operatorname{Sym}^k V$ as $\{v_{(i,j)} | i + j = k\}$, which we ordered as $v_{(k,0)} < v_{(k-1,1)} < \ldots < v_{(0,k)}$. The respective actions of σ, τ on $v_{(i,j)}$ are

$$\begin{aligned}
\sigma v_{(i,j)} &= \sigma(e_1 \cdot \ldots \cdot e_1 \cdot e_2 \cdot \ldots e_2) \\
&= \sigma e_1 \cdot \ldots \cdot \sigma e_1 \cdot \sigma e_2 \cdot \ldots \sigma e_2 \\
&= e_2 \cdot \ldots \cdot e_2 \cdot e_1 \cdot \ldots \cdot e_1 \\
&= e_1 \cdot \ldots \cdot e_1 \cdot e_2 \cdot \ldots \cdot e_2 \\
&= v_{(j,i)} \\
\tau v_{(i,j)} &= \tau(e_1 \cdot \ldots \cdot e_1 \cdot e_2 \cdot \ldots e_2) \\
&= \tau e_1 \cdot \ldots \cdot \tau e_1 \cdot \tau e_2 \cdot \ldots \tau e_2 \\
&= \omega e_1 \cdot \ldots \cdot \omega e_1 \cdot \omega e_2 \cdot \ldots \omega e_2 \\
&= \omega^{i+2j} v_{(i,j)}
\end{aligned}$$

Viewing σ, τ as elements of $\mathrm{GL}(\operatorname{Sym}^k V)$, their matrices are

$$\sigma = \mathrm{Id}^T = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} \omega^k & 0 & 0 & \ldots & 0 \\ 0 & \omega^{k+1} & 0 & \ldots & 0 \\ 0 & 0 & \omega^{k+2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \omega^{2k} \end{pmatrix}$$

Note that these are $(k+1) \times (k+1)$ matrices. Thus

$$\chi_{\operatorname{Sym}^k V}(\sigma) = \operatorname{tr} \sigma = \begin{cases} 1 & k \equiv 0 \bmod 2 \\ 0 & k \equiv 1 \bmod 2 \end{cases}$$

$$\chi_{\operatorname{Sym}^k V}(\tau) = \operatorname{tr} \tau = \sum_{j=0}^{k} \omega^{k+j} = \omega^k \sum_{j=0}^{k} \omega^j = \omega^k (1 + \omega + \omega^2 + 1 + \omega + \omega^2 + \ldots + \omega^k)$$

Note that $1 + \omega + \omega^2 = 0$, so

$$\chi_{\operatorname{Sym}^k V}(\tau) = \begin{cases} \omega^k & k \equiv 0 \bmod 3 \\ \omega^k(1+\omega) & k \equiv 1 \bmod 3 \\ 0 & k \equiv 2 \bmod 3 \end{cases}$$

When $k \equiv 0 \bmod 3$, $\omega^k = 1$. We have $1+\omega = -\omega^2$, so the case $k \equiv 1 \bmod 3$ becomes $-\omega^{k+2}$, which is -1. Thus

$$\chi_{\mathrm{Sym}^k V}(\tau) = \begin{cases} 1 & k \equiv 0 \bmod 3 \\ -1 & k \equiv 1 \bmod 3 \\ 0 & k \equiv 2 \bmod 3 \end{cases}$$

□

Proposition 0.6 (Exercise 3b, part one). *Let V be the standard, 2-dimensional, irreducible representation of S_3, and let R be the regular representation of S_3. Then*

$$\mathrm{Sym}^{k+6} V \cong \mathrm{Sym}^k V \oplus R$$

(This is an isomorphism of S_3-representations.) Note that as a consequence of the next result, $R \cong \mathrm{Sym}^5 V$, so we can also write this formula as

$$\mathrm{Sym}^{k+6} V \cong \mathrm{Sym}^k V \oplus \mathrm{Sym}^5 V$$

Proof. We will show that they have the same character, using the previous lemma. Recall that $\dim \mathrm{Sym}^k V = k+1$ and $\dim R = |S_3| = 6$, so

$$\chi_{\mathrm{Sym}^{k+6} V}(1) = \dim \mathrm{Sym}^{k+6} V = k+7 = \dim \mathrm{Sym}^k V + \dim R = \chi_{\mathrm{Sym}^k V \oplus R}(1)$$

By Lemma 0.3, $\chi_R = 0$ except at the identity, so now we just need to show that $\chi_{\mathrm{Sym}^{k+6} V} = \chi_{\mathrm{Sym}^k V}$ on σ, τ. By the previous lemma, the value on σ depends on $k \bmod 2$, and the value on τ depends on $k \bmod 3$, but $k \equiv (k+6) \bmod 3$ and $k \equiv (k+6) \bmod 2$, so they agree on σ and τ. Thus they have the same character, so they are isomorphic. (For proof that $R \cong \mathrm{Sym}^5 V$, see the next proposition.) □

Proposition 0.7 (for Exercise 3b). *Let V be the standard, 2-dimensional, irreducible representation of S_3. Then*

$$\begin{aligned} \mathrm{Sym}^0 V &\cong U \\ \mathrm{Sym}^1 V &\cong V \\ \mathrm{Sym}^2 V &\cong U \oplus V \\ \mathrm{Sym}^3 V &\cong U \oplus U' \oplus V \\ \mathrm{Sym}^4 V &\cong U \oplus V \oplus V \\ \mathrm{Sym}^5 V &\cong U \oplus U' \oplus V \oplus V \end{aligned}$$

If we write $k = 6q + r$ where $q, r \in \mathbb{Z}$ and $0 \le r \le 5$, then

$$\mathrm{Sym}^k V \cong \mathrm{Sym}^r V \oplus \left(\bigoplus R^{\oplus q} \right)$$

where $R \cong U \oplus U' \oplus V \oplus V$ is the regular representation.

Proof. We'll be somewhat slopping in writing a character of a representation of S_3 as a 3-tuple (x, y, z). For example, $\chi_U = (1, 1, 1), \chi_{U'} = (1, -1, 1), \chi_V = (2, 0, -1)$. By the Lemma 0.5, $\chi_{\mathrm{Sym}^k V} = (k + 1, \alpha, \beta)$ where

$$\alpha = \begin{cases} 1 & k \equiv 0 \bmod 2 \\ 0 & k \equiv 1 \bmod 2 \end{cases} \qquad \beta = \begin{cases} 1 & k \equiv 0 \bmod 3 \\ -1 & k \equiv 1 \bmod 3 \\ 0 & k \equiv 2 \bmod 3 \end{cases}$$

Using Proposition 2.1 of Fulton & Harris, the character of $U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$ is

$$a(1, 1, 1) + b(1, -1, 1) + c(2, 0, -1) = (k + 1, \alpha, \beta)$$

which gives the equations

$$`a + b + 2c = k + 1 \qquad a - b = \alpha \qquad a + b - c = \beta$$

We can solve these equations to get

$$\begin{aligned} a &= \frac{1}{2}(k + 1 + \alpha) - \frac{1}{3}(k + 1 - \beta) \\ b &= \frac{1}{2}(k + 1 - \alpha) - \frac{1}{3}(k + 1 - \beta) \\ c &= \frac{1}{3}(k + 1 - \beta) \end{aligned}$$

Note that these are always integers. When $k = 0$, we recover $(a, b, c) = (1, 0, 0)$, which confirms $\mathrm{Sym}^0 V = U$. When $k = 1$, we get $(a, b, c) = (0, 1, 1)$, which confirms $\mathrm{Sym}^1 V = V$. Putting this all in a table, we get

k	a	b	c
0	1	0	0
1	0	0	1
2	1	0	1
3	1	1	1
4	1	0	2
5	1	1	2

From this table, we can read off the irreducible decompositions of $\mathrm{Sym}^k V$ for $k = 0, 1, 2, 3, 4, 5$.

$$\begin{aligned} \mathrm{Sym}^0 V &= U \\ \mathrm{Sym}^1 V &= V \\ \mathrm{Sym}^2 V &= U \oplus V \\ \mathrm{Sym}^3 V &= U \oplus U' \oplus V \\ \mathrm{Sym}^4 V &= U \oplus V \oplus V \\ \mathrm{Sym}^5 V &= U \oplus U' \oplus V \oplus V \end{aligned}$$

The last statement of the claim is just induction using the previous proposition. We know that we can "peel off' multiples of 6 by taking the direct sum with a copy of R, so the result follows. □

Proposition 0.8 (Exercise 4). *Let V be the standard, 2-dimensional, irreducible representation of S_3. Then* $\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3(\mathrm{Sym}^2 V)$.

Proof. Using Exercise 1.10 of Fulton & Harris, we have an (ordered) basis $e_1 = (\omega, 1, \omega^2), e_2 = (1, \omega, \omega^2)$ with

$$\tau e_1 = \omega e_1 \qquad \tau e_2 = \omega^2 e_2 \qquad \sigma e_1 = e_2 \qquad \sigma e_2 = e_1$$

where $\sigma = (1\ 2), \tau = (1\ 2\ 3)$ and $\omega = e^{2\pi i/3}$. (Recall that σ, τ generate S_3, so this fully determines the action of S_3.) Define $v_{ij} = e_i \cdot e_j$ and $v_{ijk} = e_i \cdot e_j \cdot e_k$. Then $\{v_{11}, v_{12}, v_{22}\}$ is a basis for $\mathrm{Sym}^2 V$ and $\{v_{111}, v_{112}, v_{122}, v_{222}\}$ is a basis for $\mathrm{Sym}^3 V$. A basis for $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ is given by

$$\begin{array}{cccc} v_{111} \cdot v_{111} & v_{111} \cdot v_{112} & v_{111} \cdot v_{122} & v_{111} \cdot v_{222} \\ & v_{112} \cdot v_{112} & v_{112} \cdot v_{122} & v_{112} \cdot v_{222} \\ & & v_{122} \cdot v_{122} & v_{122} \cdot v_{222} \\ & & & v_{222} \cdot v_{222} \end{array}$$

and a basis for $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ is given by

$$\begin{array}{cccc} v_{11} \cdot v_{11} \cdot v_{11} & v_{11} \cdot v_{11} \cdot v_{12} & v_{11} \cdot v_{11} \cdot v_{22} & \\ v_{11} \cdot v_{12} \cdot v_{12} & v_{11} \cdot v_{12} \cdot v_{22} & v_{11} \cdot v_{22} \cdot v_{22} & \\ v_{12} \cdot v_{12} \cdot v_{12} & v_{12} \cdot v_{12} \cdot v_{22} & v_{12} \cdot v_{22} \cdot v_{22} & v_{22} \cdot v_{22} \cdot v_{22} \end{array}$$

We define a map, which we will show is an isomorpism.

$$\begin{aligned} \Phi : \mathrm{Sym}^2(\mathrm{Sym}^3 V) &\to \mathrm{Sym}^3(\mathrm{Sym}^2 V) \\ v_{111} \cdot v_{222} &\mapsto v_{11} \cdot v_{12} \cdot v_{22} \\ v_{112} \cdot v_{122} &\mapsto v_{12} \cdot v_{12} \cdot v_{12} \\ v_{111} \cdot v_{111} &\mapsto v_{11} \cdot v_{11} \cdot v_{11} \\ v_{222} \cdot v_{222} &\mapsto v_{22} \cdot v_{22} \cdot v_{22} \\ v_{111} \cdot v_{112} &\mapsto v_{11} \cdot v_{11} \cdot v_{12} \\ v_{122} \cdot v_{222} &\mapsto v_{12} \cdot v_{22} \cdot v_{22} \\ v_{111} \cdot v_{122} &\mapsto v_{11} \cdot v_{12} \cdot v_{12} \\ v_{112} \cdot v_{222} &\mapsto v_{12} \cdot v_{12} \cdot v_{22} \\ v_{112} \cdot v_{112} &\mapsto v_{11} \cdot v_{11} \cdot v_{22} \\ v_{122} \cdot v_{122} &\mapsto v_{11} \cdot v_{22} \cdot v_{22} \end{aligned}$$

Since we have mapped a basis to a basis, Φ is an isomorphism of vector spaces, so it just remains to check that it is equivariant with respect to the S_3 action. Since S_3 is generated by σ, τ, it is sufficient to check that Φ respects the action of σ and τ.

On the basis e_1, e_2 of V, σ acts by the permutation $(e_1\ e_2)$. Recall that the G action on the symmetric product is defined by $g(x \cdot y) = gx \cdot gy$. So σ acting on $v_{ijk} = e_i \cdot e_j \cdot e_k$, we just turn 1's into 2's and vice versa. For example, $\sigma v_{112} = v_{221} = v_{122}$. So we see that $v_{111} \cdot v_{222}$ and $v_{112} \cdot v_{122}$ are acted on by σ as identity, as are $\Phi(v_{111} \cdot v_{222})$ and $\Phi(v_{112} \cdot v_{122})$. The remaining 8 elements of each basis are acted on by σ in pairs of transpositions, and it is straightforward to check Φ commutes with σ for each basis element. For example,

$$\begin{array}{ccc} v_{111} \cdot v_{112} & \xrightarrow{\Phi} & v_{11} \cdot v_{11} \cdot v_{12} \\ \big\downarrow \sigma & & \big\downarrow \sigma \\ v_{122} \cdot v_{222} & \xrightarrow{\Phi} & v_{12} \cdot v_{22} \cdot v_{22} \end{array}$$

Now we just need to check that Φ commutes with τ. Notice that e_1, e_2 are eigenvectors for τ; more specifically, $\tau e_i = \omega^i e_i$. This implies that $\tau v_{ij} = \omega^{i+j} v_{ij}$ and $\tau v_{ijk} = \omega^{i+j+k} v_{ijk}$. Similarly, $\tau(v_{ijk} \cdot v_{mnl}) = \omega^{i+j+k+m+n+l} v_{ijk} \cdot v_{mnl}$. So to see that Φ commutes with τ, we just need to check that Φ preserves the sum of the subscripts. This can be confirmed by simply looking at the table definition of Φ. Thus we have shown that Φ commutes with σ, τ, so it is S_3-equivarient, so it is an isomorphism of representations. □

Lemma 0.9 (for Exercise 5)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$. There exists a Hermitian inner product $H : V \times V \to V$.*

Proof. Fix a basis $\{v_1, \dots, v_n\}$ of V. Then we have a vector space isomorphism

$$\phi : V \to \mathbb{C}^n \qquad v_i \mapsto e_i = (\dots, 0, 1, 0, \dots)$$

We have the classical Hermitian inner product on $\mathbb{C}^n$ given by

$$\widetilde{H} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \qquad \widetilde{H}(z, w) = \sum_{i=1}^{n} z_i \overline{w}_i$$

where $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n)$. Then define $H : V \times V \to V$ by

$$H(u, v) = \widetilde{H}(\phi v, \phi w)$$

Then H inherits all properties of $\widetilde{H}$, so H is a Hermitian inner product. □

Lemma 0.10 (for Exercise 5)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$ and let G be a finite group with representation $G \times V \to V$. Suppose $\widetilde{H} : V \times V \to \mathbb{C}$ is a Hermitian inner product. Define*

$$H : V \times V \to \mathbb{C} \qquad H(u, v) = \frac{1}{|G|} \sum_{g \in G} \widetilde{H}(gu, gv)$$

Then H is a G-invariant Hermitian inner product on V.

Proof. It is straightforward to show via computation that H is additive, linear in the first entry, antilinear in the second entry, conjugate-symmetric, and positive-definite. We will show that H is G-invariant. Let $x \in G$.

$$H(u, v) = \frac{1}{|G|} \sum_{g \in G} \widetilde{H}(gu, gv)$$

$$H(xu, xv) = \frac{1}{|G|} \sum_{g \in G} \widetilde{H}(gxu, gxv)$$

As g runs over each element in G, so does xg for a fixed $x \in G$, since $G \to G, g \mapsto xg$ is a permutation of G. Thus these two sums are the same up to permutation summands. Hence $H(u, v) = H(xu, xv)$, and H is G-invariant. □

Proposition 0.11 (Exercise 5)**.** *Let V be an irreducible representation of a finite group G. Up to scalars, there is a unique Hermitian inner product on V that is G-invariant.*

Proof. By the previous two lemmas, a G-invariant Hermitian inner product on V exists. Suppose H, H' are two such products. Define

$$\begin{aligned} \widetilde{H} : V \to V^* \qquad & v \mapsto \Big(u \mapsto H(v,u)\Big) \\ \widetilde{H}' : V \to V^* \qquad & v \mapsto \Big(u \mapsto H'(v,u)\Big) \end{aligned}$$

Since H, H' are nondegenerate forms, $\widetilde{H}, \widetilde{H}'$ are vector space isomorphisms. Then the composition $(\widetilde{H}')^{-1} \circ \widetilde{H} : V \to V$ is a vector space automorphism. Since $\widetilde{H}'$ is G-linear, its inverse is also G-linear, so this is a composition of G-linear maps, which is therefore G-linear. Then by Schur's Lemma, this composition must be equal to λI for some $\lambda \in \mathbb{C}$. This implies that $\widetilde{H}' = \lambda \widetilde{H}$, which implies that $H' = \lambda H$. □

Lemma 0.12 (for Exercises 6,7)**.** *Let V be a finite dimensional representation of G, and let $g \in G$ have eigenvalues $\{\lambda_i\}_{i=1}^{\dim V}$ (viewing $g \in \mathrm{GL}(V)$). Then the eigenvalues of g as an automorphism of $V^{\otimes n}$ are*

$$\prod_{k=1}^{n} \lambda_{i_k}$$

where $i_1, i_2, \ldots, i_n \in \{1, \ldots, \dim V\}$. Viewing g as an automorphism of $\mathrm{Sym}^n V$, the eigenvalues are all such products with $i_1 \le i_2 \le \ldots \le i_n$. Viewing g as an automorphism of $\wedge^n V$, the eigenvalues are all such products with $i_1 < i_2 < \ldots < i_n$.

Proof. Define $I = \{1, \ldots, \dim V\}$, and let $w_i \in V$ be the corresponding eigenvector for λ_i. Let $i_1, i_2, \ldots, i_n \in I$. Viewing g as $g \in \mathrm{GL}(V^{\otimes n})$.

$$\begin{aligned} g\left(w_{i_1} \otimes w_{i_2} \otimes \ldots \otimes w_{i_n}\right) &= g w_{i_1} \otimes g w_{i_2} \otimes \ldots \otimes g w_{i_n} \\ &= \lambda_{i_1} w_{i_1} \otimes \lambda_{i_2} w_{i_2} \otimes \ldots \otimes \lambda_{i_n} w_{i_n} \\ &= \left(\prod_{k=1}^{n} \lambda_{i_k}\right) (w_{i_1} \otimes w_{i_2} \otimes \ldots \otimes w_{i_n}) \end{aligned}$$

so we see that $w_{i_1} \otimes w_{i_2} \otimes \ldots \otimes w_{i_n}$ is an eigenvector of g with eigenvalue $\prod_{k=1}^{n} \lambda_{i_k}$. Thus all products of this type are eigenvalues of g on $V^{\otimes n}$. Since the dimesion of $V^{\otimes n}$ is $(\dim V)^n$, and we have found $(\dim V)^n$ eigenvalues (counting multiplicities), these are all of the eigenvalues.

Now we prove the statements about $\mathrm{Sym}^n V$ and $\wedge^n V$. Since these are subspaces of $V^{\otimes n}$, the eigenvalues must be a subset of these products. Thinking of g as $g \in \mathrm{GL}(\mathrm{Sym}^n V)$, we have identified tensors up to permutation, so we can do the same calculation to have g act on $w_{i_1} \otimes \ldots \otimes w_{i_n}$, except now we may permute them so that $i_1 \le i_2 \le \ldots \le i_n$. So in order not to count eigenvalues too many times, we restrict to products where $i_1 \le \ldots \le i_n$. A similar argument works for the statement about $\wedge^n V$. □

Proposition 0.13 (Exercise 6)**.** *Let V be a finite dimensional representation of a finite group G. Then*

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right)$$

Proof. Let $d = \dim V$, and let $\{\lambda_i\}_{i=1}^d$ be the set of eigenvalues for $g : V \to V$. Then

$$\chi_V(g) = \sum_{i=1}^{d} \lambda_i \qquad \chi_V(g^2) = \sum_i \lambda_i^2 \qquad \chi_V(g)^2 = \left(\sum_i \lambda_i\right)^2$$

By the previous lemma, the set of eigenvalues for $g : \mathrm{Sym}^2 V \to \mathrm{Sym}^2 V$ is $\{\lambda_i\lambda_j | i \le j\}$. Now, reusing the identity

$$\sum_{i<j} \lambda_i\lambda_j = \frac{\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2}{2}$$

found on page 13 of Fulton & Harris, we can evaluate $\chi_{\mathrm{Sym}^2 V}$.

$$\begin{aligned}\chi_{\mathrm{Sym}^2 V}(g) &= \sum_{i\le j} \lambda_i\lambda_j = \left(\sum_{i=j} \lambda_i\lambda_j\right) + \left(\sum_{i<j} \lambda_i\lambda_j\right) = \sum_i \lambda_i^2 + \frac{\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2}{2} \\ &= \frac{1}{2}\left(\sum_i \lambda_i^2 + \left(\sum_i \lambda_i\right)^2\right) = \frac{1}{2}\left(\chi_V(g^2) + \chi_V(g)^2\right)\end{aligned}$$

□

Proposition 0.14 (Exercise 7)**.** *Let V be a representation of G. The characters of* $\mathrm{Sym}^k V$ *and $\wedge^k V$ are*

$$\chi_{\mathrm{Sym}^k V}(g) = \sum \prod_{i=1}^{k} \frac{\chi_V(g^i)^{m_i}}{m_i! i^{m_i}} \qquad \chi_{\wedge^k V}(g) = \sum \prod_{i=1}^{k} \frac{\chi_V(g^i)^{m_i}}{m_i! i^{m_i}}$$

where both sums are over multi-indices $(m_1, \ldots, m_k)$ where $\sum_j j m_j = k$ and $m_j \ge 0$.

Proof. Let $d = \dim V$ and let $\{\lambda_i\}_{i=1}^d$ be the set of eigenvalues for $g \in \mathrm{GL}(V)$. Let h_k be the complete homogenous symmetric polynomial, e_k be the elementary symmetric polynomial, and p_k be the power sum symmetric polynomial, all in $\lambda_1, \ldots, \lambda_d$. Concretely, they are

$$h_k = \sum_{i_1 \le \ldots i_m} \lambda_{i_1} \ldots \lambda_{i_m} \qquad e_k = \sum_{i_1 < \ldots < i_m} \lambda_{i_1} \ldots \lambda_{i_m} \qquad p_k = \sum_i \lambda_i^k$$

Note that $\chi_V(g^k) = p_k$. By Lemma 0.12, the eigenvalues for $g \in \mathrm{GL}(\mathrm{Sym}^k V)$ are products $\lambda_{i_1} \ldots \lambda_{i_m}$ where $i_1 \le \ldots \le i_m$, so the trace is the sum over all such products. Similarly for $\wedge^k V$, the eigenvalues are the same kind of products where $i_1 < \ldots < i_m$, so the character is the sum over those products. Thus

$$\chi_{\mathrm{Sym}^k V}(g) = h_k \qquad \chi_{\wedge^k V}(g) = e_k$$

Using Newton's Identities, we can write both h_k and e_k in terms of the polynomials $p_1, \ldots, p_k$. That is to say, we can write both $\chi_{\mathrm{Sym}^k V}(g)$ and $\chi_{\wedge^k V}(g)$ in terms of $\chi_V(g), \chi_V(g^2), \ldots, \chi_V(g^k)$.

$$h_k = \sum \prod_{i=1}^{k} \frac{p_i^{m_i}}{m_i! i^{m_i}} \qquad e_k = \sum \prod_{i=1}^{k} \frac{p_i^{m_i}}{m_i! i^{m_i}}$$

where both sums are over multi-indices $(m_1, \ldots, m_k)$ where $\sum_j jm_j = k$ and $m_j \geq 0$. Changing this into the notation of characters, we get exactly the claimed formulas.

$$\chi_{\mathrm{Sym}^k V}(g) = \sum \prod_{i=1}^{k} \frac{\chi_V(g^i)^{m_i}}{m_i! i^{m_i}} \qquad \chi_{\wedge^k V}(g) = \sum \prod_{i=1}^{k} \frac{\chi_V(g^i)^{m_i}}{m_i! i^{m_i}}$$

□

Proposition 0.15 (Exercise 8). *Let G be a finite group acting on a finite set X. Let*

$$V = \bigoplus_{x \in X} \mathbb{C}x$$

be the permutation representation of G. Then for $g \in G$, $\chi_V(g)$ is the number of elements of X fixed by g.

Proof. Recall that the action of G on V is defined by

$$G \times V \to V \qquad (g, x) \mapsto g \cdot x$$

where $\cdot$ is the action of G on X. Then we extend linearly to all of V. That is, each $g \in G$ acts on V to permute the basis X of V. Thus, the matrix representation of g in the basis X is a permutation matrix. In this permutation matrix, a one along the diagonal represents an element $x \in X$ for which $g \cdot x = x$, that is, a fixed point of g. Since g is represented by a permutation matrix, all entries are zero except for a single one in each row, so the trace is the number of fixed points. □

Proposition 0.16 (Exercise 9). *Let V, W be irreducible representations of a finite group G, and $L_0 : V \to W$ a linear map. Define $L : V \to W$ by*

$$L = \frac{1}{|G|} \sum_{g \in G} g^{-1} L_0 g$$

or more explicitly,

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot v)$$

Then L is a G-module homomorphism. Consequently, if V and W are not isomorphic representations, then $L = 0$, and if $V = W$, then L is multiplication by $\frac{\mathrm{tr}(L_0)}{\dim V}$.

Proof. We claim that L is a G-module homomorphism. Let $h \in G$.

$$L(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot (h \cdot v)) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0((gh) \cdot v)$$

Define $x = gh$. Then $g^{-1} = hx^{-1}$. Note that as g ranges over G, so does x, so we can rewrite our sum as

$$\begin{aligned} \frac{1}{|G|} \sum_{x \in G} (hx^{-1}) \cdot L_0(x \cdot v) &= \frac{1}{|G|} \sum_{x \in G} h \cdot (x^{-1} \cdot L_0(x \cdot v)) \\ &= h \cdot \frac{1}{|G|} \sum_{x \in G} x^{-1} \cdot L_0(x \cdot v) \\ &= h \cdot L(v) \end{aligned}$$

Thus L is G-linear. Now by Schur's Lemma, since $L : V \to W$ is a G-module homomorphism between irreducible representations, it is either an isomorphism or zero. Hence if V, W are not isomorphic, $L = 0$. If $V = W$, Schur's Lemma tells us that $L = \lambda I$ for some λ. We know that $\lambda = \frac{\operatorname{tr}(L)}{\dim V}$. Since tr is linear,

$$\operatorname{tr}(L) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(g^{-1}L_0 g) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(L_0) = \operatorname{tr}(L_0)$$

since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ (we applied this where $A = g^{-1}L_0, B = g$). Thus $\lambda = \frac{\operatorname{tr}(L_0)}{\dim V}$. □

Lemma 0.17 (for Exercise 10)**.** *Let G be a finite group with irreducible representations $\rho_V : V \to \mathrm{GL}(V)$ and $\rho_W : W \to \mathrm{GL}(W)$. For $g \in G$, we view $\rho_V(g)$ as a matrix with ij-th entry $\alpha_{ij}(g)$. Similarly, let β_{ij} be the ij-th entry of $\rho_W(g)$. If $V \not\cong W$, then*

$$\frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\beta_{\ell j}(g) = 0$$

If the matrices $\rho_V(g)$ are unitary, then for all i, j, k, ℓ we have

$$(\alpha_{ik}, \beta_{\ell j}) = 0$$

(For the case $V = W$, see next lemma.)

Proof. Let $L_0 : V \to W$ be a linear map, viewed as a matrix, with ij-th entry L^0_{ij}. By the previous result,

$$L = \frac{1}{|G|}\sum_{g\in G}\rho_W(g^{-1})\circ L_0 \circ \rho_V(g)$$

is the zero map. On a matrix-entry level, we have

$$\begin{aligned} 0 &= L_{ij} \\ &= \frac{1}{|G|}\sum_{g\in G}\Big(\rho_W(g^{-1})\circ L_0\circ\rho_V(g)\Big)_{ij} \\ &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{k,\ell}\left(\rho_W(g^{-1})_{ik}\right)L^0_{k\ell}\left(\rho_V(g)\right)_{\ell j}\right) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{k,\ell}\Big(\alpha_{ik}(g^{-1})\Big)L^0_{k\ell}\Big(\beta_{\ell j}(g)\Big) \end{aligned}$$

Since L_0 was any linear map, we can choose $L^0_{k\ell}$ to be anything in $\mathbb{C}$, and this equality will still hold. In particular, we can choose $L^0_{k\ell}$ to be zero for everything except one fixed pair (k_0, ℓ_0), and obtain

$$0 = \frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\beta_{\ell j}(g)$$

where $k = k_0, \ell = \ell_0$. Now suppose $\rho_V(g)$ is unitary. Then $\alpha_{ij}(g^{-1}) = \overline{\alpha_{ij}(g)}$, so we get

$$(\alpha_{ik}, \beta_{\ell j}) = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha_{ik}(g)}\beta_{\ell j}(g) = \frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\beta_{\ell j}(g) = 0$$

□

Lemma 0.18 (for Exercise 10). *Let G be a finite group with an irreducible representation $\rho : V \to \mathrm{GL}(V)$. For $g \in G$, we view $\rho(g)$ as a matrix with ij-th entry $\alpha_{ij}(g)$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\alpha_{\ell j}(g) = \frac{1}{\dim V}\delta_{k\ell}\delta_{ij}$$

If the matrices $\rho(g)$ are unitary, then

$$(\alpha_{ki},\alpha_{\ell j}) = \frac{1}{\dim V}\delta_{k\ell}\delta_{ij}$$

Proof. Let $L_0 : V \to V$ be a linear map, viewed as a matrix, with ij-th entry L^0_{ij}. By the previous result,

$$L = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\circ L_0 \circ \rho(g)$$

is multiplication by $\lambda = \frac{\mathrm{tr}(L_0)}{\dim V}$. Denoting the identity map $V \to V$ by I, we can write the statement $L = \lambda I$ as

$$L = \frac{1}{\dim V}\sum_k L^0_{kk} I = \frac{1}{\dim V}\sum_{k,\ell}\delta_{k\ell}L^0_{k\ell}I$$

On a matrix-entry level, this says

$$L_{ij} = \frac{1}{\dim V}\sum_{k,\ell}\delta_{k\ell}L^0_{k\ell}I_{ij} = \frac{1}{\dim V}\sum_{k,\ell}\delta_{k\ell}\delta_{ij}L^0_{k\ell}$$

On the other hand, we can also write L_{ij} using the definition of L in the following way.

$$\begin{aligned} L_{ij} &= \frac{1}{|G|}\sum_{g\in G}\Big(\rho(g^{-1})\circ L_0\rho(g)\Big)_{ij} \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{k,\ell}\Big(\alpha_{ik}(g^{-1})\Big)L^0_{k\ell}\Big(\alpha_{\ell j}(g)\Big) \\ &= \frac{1}{|G|}\sum_{k,\ell}\left(\sum_{g\in G}\Big(\alpha_{ik}(g^{-1})\alpha_{\ell j}(g)\Big)\right)L^0_{k\ell} \end{aligned}$$

As in the proof of the previous lemma, L_0 was arbitrary, so this equality holds for any values of $L^0_{k\ell}$. So if we choose them to be zero everywhere except for some pair (k_0,ℓ_0), the sums over k,ℓ collapse to a single term, both divisible by $L^0_{k_0\ell_0}$, which cancel. All this to say, we can equate the coefficients of $L^0_{k\ell}$ in these sums to obtain

$$\frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\alpha_{\ell j}(g) = \frac{1}{\dim V}\delta_{k\ell}\delta_{ij}$$

Now suppose the matrices $\rho(g) = (\alpha_{ik}(g))$ are unitary. Then $\alpha_{ik}(g^{-1}) = \overline{\alpha_{ki}(g)}$, so we get

$$(\alpha_{ki},\alpha_{\ell j}) = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha_{ki}(g)}\alpha_{\ell j} = \frac{1}{|G|}\sum_{g\in G}\alpha_{ik}(g^{-1})\alpha_{\ell j}(g) = \frac{1}{\dim V}\delta_{k\ell}\delta_{ij}$$

□

Proposition 0.19 (Exercise 10)**.** *Let G be a finite group, and let $\rho_m : G \to \mathrm{GL}(V_m)$ be the set of irreducible representations. Suppose that for $g \in G$, $\rho_m(g)$ is a unitary matrix. For $g \in G$, viewing $\rho_m(g)$ as a matrix, let the ij-th entry be $\alpha_{ij}^m(g)$. Then the set of all α_{ij}^m forms an orthogonal basis for the vector space of functions $G \to \mathbb{C}$ with inner product given by*

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Proof. Note that the space of functions $G \to \mathbb{C}$ has dimension $|G|$, and the set $\left\{\alpha_{ij}^m\right\}$ has size

$$\sum_m (\dim V_m)^2$$

which is equal to $|G|$, so it is at least possible that these functions form a basis. First, suppose $m \neq n$ and consider the functions α_{ik}^m and $\alpha_{\ell j}^n$. Since $m \neq n$, $V_m \not\cong V_n$, so by Lemma 0.17, we have

$$\left(\alpha_{ik}^m, \alpha_{\ell j}^n\right) = 0$$

If $m = n$, then by Lemma 0.18,

$$\left(\alpha_{ik}^m, \alpha_{\ell j}^m\right) = \frac{1}{\dim V_m} \delta_{i\ell} \delta_{kj}$$

That is, the inner product is zero unless $i = \ell$ and $k = j$, in which case $\alpha_{ik}^m = \alpha_{\ell j}^m$. Thus

$$\left(\alpha_{ik}^m, \alpha_{\ell j}^n\right) = \begin{cases} \frac{1}{\dim V^m} & m = n, i = \ell, k = j \\ 0 & \text{else} \end{cases}$$

which is to say that $\left\{\alpha_{ij}^m\right\}$ is an orthogonal basis. ☐

Lemma 0.20 (for Exercise 11)**.** *The elements of* $\mathrm{SL}_2(\mathbb{Z}/3)$ *are*

$$\begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} & \begin{pmatrix} 2 & x \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ x & 2 \end{pmatrix} \\ \begin{pmatrix} x & 2 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 1 & x \end{pmatrix} & \begin{pmatrix} x & 1 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 2 & x \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \end{matrix}$$

where x can be 1 or 2. (This gives a total of 24 elements.)

Proof. There can be no elements of $\mathrm{SL}_2(\mathbb{Z}/3)$ with three or four zero entries, since that would force the determinant to be zero. If there are two zero entries, since it is invertible, the zeroes occur along a diagonal. The nonzero diagonal must have product 1 if it is the main diagonal, or $-1 = 2$ if it is the antidiagonal. Thus the elements of $\mathrm{SL}_2(\mathbb{Z}/3)$ with two zero entries are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Now consider elements with exactly one zero entry. As before, the nonzero diagonal must have product 1 if it is the main diagonal and product 2 if is the antidiagonal. So all of the elements with exactly one zero look like

$$\begin{matrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} & \begin{pmatrix} 2 & x \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ x & 2 \end{pmatrix} \\ \begin{pmatrix} x & 2 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 1 & x \end{pmatrix} & \begin{pmatrix} x & 1 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 2 & x \end{pmatrix} \end{matrix}$$

for $x = 1$ or $x = 2$. Finally, consider elements with no zero entries.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/3)$$

We know $ad - bc = 1$. Since none of a, b, c, d are zero, $ad = 1$ or $ad = 2$. Since $bc \neq 0$, $ad \neq 1$. Thus $ad = 2$, and $bc = 1$. This means that $\{a, d\} = \{1, 2\}$ and b, c can either both be 1 or both be 2. So the elements of this type are

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

□

Lemma 0.21 (for Exercise 11)**.** *The conjugacy classes of* $\mathrm{SL}_2(\mathbb{Z}/3)$ *are*

$$C_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$C_1 = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

$$C_2 = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

$$C_3 = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \right\}$$

$$C_4 = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\}$$

$$C_5 = \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$$

$$C_6 = \left\{ \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

The orders of the elements in each class are given in the following table.

Class	*Order*
C_0	*1*
C_1	*2*
C_2	*4*
C_3	*6*
C_4	*6*
C_5	*3*
C_6	*3*

Proposition 0.22 (Exercise 11)**.** *Let* $\omega = e^{2\pi i/3}$*. The character table of* $\mathrm{SL}_2(\mathbb{Z}/3)$ *is*

(size)	1	1	6	4	4	4	4
$\mathrm{SL}_2(\mathbb{Z}/3)$	C_0	C_1	C_2	C_3	C_4	C_5	C_6
U	1	1	1	1	1	1	1
W_1	1	1	1	ω^2	ω	ω	ω^2
W_2	1	1	1	ω	ω^2	ω^2	ω
V	2	-2	0	1	1	-1	-1
$V \otimes W_1$	2	-2	0	ω^2	ω	$-\omega$	$-\omega^2$
$V \otimes W_2$	2	-2	0	ω	ω^2	$-\omega^2$	$-\omega$
T	3	3	-2	0	0	0	0

Proof. We can deduce the dimensions of the irreducible representations using Corollary 2.18 of Fulton & Harris. We have 7 positive numbers for which the sum of squares is 24. None can be 4, since then the dimensions of the remaining irreps would all have to be 1, but $16+1+1+1+1+1+1 \neq 24$. There also can't be two dim=3 irreps, since $9+9 = 18$. There can't be zero dim=3 irreps, because we can't get 24 by a sum of 4's and 1's with 7 terms. Thus there is exactly one dim=3 irrep. We have 6 remaining irreps. Let a be the number of dim=1 irreps and b be the number of dim=2 irreps. Then $a + b = 6$ and $a + 4b = 15$. This has a unique solution $(a, b) = (3, 3)$. Thus the dimensions of the irreps are $(1, 1, 1, 2, 2, 2, 3)$. We always have the trivial representation, so at this point our character table looks like

(size)	1	1	6	4	4	4	4
$\mathrm{SL}_2(\mathbb{Z}/3)$	C_0	C_1	C_2	C_3	C_4	C_5	C_6
U	1	1	1	1	1	1	1
	1						
	1						
	2						
	2						
	2						
	3						

First let's find the dim=1 irreps. These are homomorphisms $\mathrm{SL}_2(\mathbb{Z}/3) \to \mathbb{C}^*$. An element of order n must be sent to an nth root of unity. It turns out (not obviously) that it's impossible to send C_1 to -1, so it has to go to 1. Let $\omega = e^{2\pi i/3}$, and define W_1, W_2 by the following characters.

(size)	1	1	6	4	4	4	4
$\mathrm{SL}_2(\mathbb{Z}/3)$	C_0	C_1	C_2	C_3	C_4	C_5	C_6
U	1	1	1	1	1	1	1
W_1	1	1	1	ω^2	ω	ω	ω^2
W_2	1	1	1	ω	ω^2	ω^2	ω

One can check that these are in fact representations, and that $(\chi_{W_1}, \chi_{W_1}) = (\chi_{W_2}, \chi_{W_2}) = 1$, so these are in fact irreps. We need to find a character of a two dimensional irreducible representation. Notice that $H = C_0 \cup C_1 \cup C_2$ is an index 3 subgroup of $\mathrm{SL}_2(\mathbb{Z}/3)$ isomorphic to the quaternion group. The character table of the quaternion group is

	1	1	2	2	2
H	1	-1	i	j	k
U_H	1	1	1	1	1
A_i	1	1	1	-1	-1
A_j	1	1	-1	1	-1
A_k	1	1	-1	-1	1
V_H	2	-2	0	0	0

Note that C_0 corresponds to the conjugacy class of 1, C_1 corresponds to the conjugacy class of -1, and the conjugacy classes of i, j, k make up C_2. To distinguish U as a representation of $H, G = \mathrm{SL}_2(\mathbb{Z}/3)$, write U_H and U_G for the H- and G-representations respectively. Note that $\operatorname{Res} U_G = \operatorname{Res} W_1 = \operatorname{Res} W_2 = U_H$. Using Frobenius reciprocity, we can calculate

$$(\chi_{\operatorname{Ind} A_i}, \chi_{U_G})_G = (\chi_{\operatorname{Ind} A_i}, \chi_{W_1})_G = (\chi_{\operatorname{Ind} A_i}, \chi_{W_2})_G = (\chi_{A_i}, \chi_{U_H})_H = 0$$

so $\operatorname{Ind} A_i$ does not include any direct summands of U_G, W_1, or W_2. Since it is three dimensional, it must be the one three dimensional irreducible represntation of G, which at this point we know nothing about, except that it exists. Using the Mackey formula

$$\chi_{\operatorname{Ind} W}(g) = \frac{1}{|H|} \sum_{x \in G, x^{-1}gx \in H} \chi_W(x^{-1}gx)$$

for the character of an induced representation, we can compute $\chi_{\operatorname{Ind} A_i} = (3, 3, -2, 0, 0, 0, 0)$, which gives us the character of the mystery dim=3 irrep. Now consider $\operatorname{Ind} V_H$. Using Frobenius reciprocity,

$$(\chi_{\operatorname{Ind} V_H}, \chi_{U_G})_G = (\chi_{\operatorname{Ind} V_H}, \chi_{W_1})_G = (\chi_{\operatorname{Ind} V_H}, \chi_{W_2})_G = (\chi_{V_H}, \chi_{U_H})_H = 0$$

so U_G, W_1, W_2 do not appear in the direct sum decomposition of $\operatorname{Ind} V_H$. Again, using the Mackey formula, we can compute that $\operatorname{Ind} V_H = (6, -6, 0, 0, 0, 0)$. Then notice that $\chi_T + \chi_T \neq \chi_{\operatorname{Ind} V_H}$, so $\operatorname{Ind} V_H \ncong T \oplus T$. Thus $\operatorname{Ind} V_H$ must be a direct sum of three of the various 2-dimension irreps of G.

Suppose it is a direct sum of three copies of the same representation V_0. Then $3\chi_{V_0} = (6, -6, 0, 0, 0, 0)$, which implies $\chi_{V_0} = (2, -2, 0, 0, 0, 0)$. However, we compute $(\chi_{V_0}, \chi_{V_0}) \neq 1$, so this is not the character of an irreducible representation.

Write $\operatorname{Ind} V_h = V_1 \oplus V_2 \oplus V_3$. The summands V_1, V_2, V_3 may be the same or different, all we know is that they are all 2-dimensional irreps of G, we just know they aren't all the same. So $\chi_{V_1} + \chi_{V_2} + \chi_{V_3} = (6, -6, 0, 0, 0, 0)$. Consider the values of χ_{V_i} on the class C_1. It must be a sum of up to square roots of unity, so the possibilities are $\{0, 1, -1, 2, -1\}$. Since we need to have three of these add up to 6, they must all be -2. So $\chi_{V_i}(C_1) = -2$. Since V_i is irreducible,

$$\begin{aligned} 0 = (\chi_{V_i}, \chi_T) &= \frac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g) \overline{\chi_T(g)} = \frac{1}{24} \left(1(2)(3) + 1(a)(3) + 6(-2)b\right) \\ &= \frac{1}{24}(6 + 3\chi_{V_i}(C_1) - 12\chi_{V_i}(C_2)) \implies 6 + 3\chi_{V_i}(C_1) = 12\chi_{V_i}(C_2) = 0 \\ &\implies \chi_{V_i}(C_2) = \frac{1}{4}(\chi_{V_i}(C_1) + 2) \end{aligned}$$

Since we know that $\chi_{V_i}(C_1) = -2$, we get $\chi_{V_i}(C_2) = 0$ for $i = 1, 2, 3$. Let $a_i = \chi_{V_i}(C_3), b_i = \chi_{V_i}(C_4), c_i = \chi_{V_i}(C_5), d_i = \chi_{V_i}(C_6)$. Since V_i is irreducible, we get

$$(\chi_{V_i}, \chi_U) = 0 \qquad (\chi_{V_i}, \chi_{W_1}) = 0 \qquad (\chi_{V_i}, \chi_{W_2}) = 0$$

expanding out and simplifying these equations, we get

$$a_i + b_i + c_i + d_i = 0 \qquad a_i + \omega b_i + \omega c_i + d = 0 \qquad \omega a_i + b + c + \omega d_i = 0$$

We can combine these equations to conclude that $c_i = -b_i$ and $d_i = -a_i$. We have reduced everything to finding the values of $a_1, a_2, a_3, b_1, b_2, b_3$. We have the equations

$$a_1 + a_2 + a_3 = 0 \qquad b_1 + b_2 + b_3 = 0$$

which can eliminate the variables a_3, b_3. Using the fact that $\left(\chi_{V_i}, \chi_{V_j}\right) = \delta_{ij}$, in the case $i \neq j$ we get

$$1 + \overline{a}_i a_j + \overline{b}_i b_j = 0$$

which is actually 3 equations. Using these, we can eliminate 3 more variables, leaving just a_1 undetermined. Doing a bit more algebra, we can finally conclude that V_1, V_2, V_3 are all distinct, and the final character table is as claimed. □

Proposition 0.23 (Exercise 12)**.** *Let $H = A_5 \subset G = S_5$. Then*

$$\operatorname{Ind} U = U \oplus U' \qquad \operatorname{Ind} V = V \oplus V' \qquad \operatorname{Ind} W = W \oplus W' \qquad \operatorname{Ind} Y = \operatorname{Ind} Z = \wedge^2 V$$

(For the character tables of S_5 and A_5 detailing these these representations, see tables below.)

Proof. Since U, V, W refer to representations of both H and G, to distinguish them we'll write U_H, V_H, W_H for the H-reps, and U_G, V_G, W_G for the G-reps. First we tackle $\operatorname{Ind} U$. Note that $\operatorname{Res} U_G = \operatorname{Res} U' = U_H$. By Frobenius reciprocity,

$$\begin{aligned} (\chi_{\operatorname{Ind} U_H}, \chi_{U_G})_G &= (\chi_{U_H}, \chi_{\operatorname{Res} U_G})_H = (\chi_{U_H}, \chi_{U_H})_H = 1 \\ (\chi_{\operatorname{Ind} U_H}, \chi_{U'})_G &= (\chi_{U_H}, \chi_{\operatorname{Res} U'})_H = (\chi_{U_H}, \chi_{U_H})_H = 1 \end{aligned}$$

so U_G and U' both in $\operatorname{Ind} U_H$ as a single direct summand. Since H is an index two subgroup, we know that $\dim \operatorname{Ind} T = 2 \dim T$ for any representation T, so $\dim \operatorname{Ind} U_H = 2 \dim U = 2$.

Thus $\operatorname{Ind} U_H = U_G \oplus U'$, since there can't be any other summands, since that would make the dimension too big. Now consider $\operatorname{Ind} V_H$. Note that $\operatorname{Res} V_G = \operatorname{Res} V' = V_H$, so again using Frobenius reciprocity,

$$(\chi_{\operatorname{Ind} V_H}, \chi_{V_G})_G = (\chi_{V_H}, \chi_{\operatorname{Res} V_G})_H = (\chi_{V_H}, \chi_{V_H})_H = 1$$
$$(\chi_{\operatorname{Ind} V_H}, \chi_{V'})_G = (\chi_{V_H}, \chi_{\operatorname{Res} V'})_H = (\chi_{V_H}, \chi_{V_H})_H = 1$$

thus $\operatorname{Ind} V_H = V_G \oplus V'$. There can't be other summands by dimension counting. Now consider $\operatorname{Ind} W_H$. Again, $\operatorname{Res} W_G = \operatorname{Res} W' = W_H$, and we perform exactly the same calculation with Frobenius reciprocity to conclude that $\operatorname{Ind} W_H \cong W_G \oplus W'$.

$$(\chi_{\operatorname{Ind} W_H}, \chi_{W_G})_G = (\chi_{W_H}, \chi_{\operatorname{Res} W_G})_H = (\chi_{W_H}, \chi_{W_H})_H = 1$$
$$(\chi_{\operatorname{Ind} W_H}, \chi_{W'})_G = (\chi_{W_H}, \chi_{\operatorname{Res} W'})_H = (\chi_{W_H}, \chi_{W_H})_H = 1$$

Finally, consider $\operatorname{Ind} Y$ and $\operatorname{Ind} Z$. We have $\operatorname{Res} \wedge^2 V = (6, 0, -2, 1, 1)$.

$$(\chi_{\operatorname{Ind} Y}, \chi_{\wedge^2 V})_G = (\chi_Y, \chi_{\operatorname{Res} \wedge^2 V})_H = 1$$
$$(\chi_{\operatorname{Ind} Z}, \chi_{\wedge^2 V})_G = (\chi_Z, \chi_{\operatorname{Res} \wedge^2 V})_H = 1$$

(Note that these, unlike the previous calculations, are not as trivial - one must work out the inner product $(\chi_Y, \chi_{\operatorname{Res} \wedge^2 V})_H$ and same for Z. But they do come out to be 1.) Thus $\operatorname{Ind} Y = \operatorname{Ind} Z = \wedge^2 V$, and there can't be other summands by dimension counting. □

The following are the character tables for S_5 and A_5. These are helpful to understand the statement of the previous proposition.

	1	20	15	12	12
A_5	1	(123)	(12)(34)	(12345)	(21345)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\wedge^2 V$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1